Some Comments on Sampling of Ergodic Process, an Ergodic Theorem and Turbulent Pressure Fluctuations

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Abstract

We present a new proof of the an Ergodic Theorem for Wide-Sense Stationary Random Processes added with a new canonical Sampling Theorem for finite time duration signals in the frequency domain (periodograms) which is free from the Nyquist interval Sampling restriction. We point out the usefulnees of such theorem in the context of a model of random vibrations transmission (pressure fluctuations).

1 Introduction

A basic problem in the applications of stochastic processes is the estimation of a signal $x(t)$ in the presence of an additive interference $f(t)$ (noise). The available information (data) is the sum $S(t) = x(t) + f(t)$ and the problem is to establish the presence of $x(t)$ or to estimate its form. The solution of this problem depends on the state of our prior knowledge concerning of the noise statistics. One of the main results on the subject is the idea of maximize the output signal-to-noise ratio ([1]), the matched filter system. One of the most important results on the subject is that if one knows a priori the signal in the frequency domain $X(w)$ and, mostly important, the frequency domain expression for the noise correlation statistics function $S_{ff}(w)$, one has, at least in the theoretical grounds, the exactly expression for the optimum

filter transference function $H_{opt}(w) = k(X^*(w))(S_{ff}(w))^{-1} \cdot e^{-iw\bar{t}}$, here $\bar{t}$ denotes a certain time on the observation interval process $[-A, A]$, supposed to be finite here.

As a consequence, it is important to have estimators and analytical expressions for the correlation function for the noise, specially in the physical situation of finite-time duration noise observation. Another very important point to be remarked is that in most of the cases of observed noise (as in the turbulence research ([2]); one should consider the noise (at least in the context of a first approximation) as an Ergodic Random Process.

We aim in this note to present in Section 2 [of a more mathematical oriented nature] a rigorous functional analytic proof of an Ergodic Theorem stating the equality of time-averages and Ensenble-averages for wide-sense mean continuous Stationary Random Processes. In Section 3 somewhat of electrical Engineering oriented nature, we present a new approach for sampling analysis of noise, which leads to canonical and invariant analytical expressions for the Ergodic noise correlation function $S_{ff}(w)$ already taking into account with the finite time-observation parameter which explicitly appears in the structure formulae, a new result on the subject, since it does not require the existence of Nyquist critical frequency on the sampling rate, besides of removing, in principle, the aliasing problem in the computer-numerical sampling evaluations. In Section 4 we present an mathematical-theoretical application of the above theoretical results for a model of Turbulent Pressure fluctuations.

2 A Rigorous Mathematical proof of the Ergodic theorem for Wide-Sense Stationary Stochastic Process

Let us start our section by considering a wide-sense mean continuous stationary real-valued process $\{X(t),\ -\infty < t < \infty\}$ in a probability space $\{\Omega,\ d\mu(\lambda),\ \lambda \in \Omega\}$. Here Ω is the event space and $d\mu(\lambda)$ is the underlying probability measure.

It is well-know ([1]) that one can always represent the above mentioned wide-sense stationary process by means of a unitary group on the Hilbert Space $\{L^2(\Omega),\ d\mu(\lambda)\}$. Namely [in the

quadratic-mean sense in Engineering jargon]

$$X(t) = U(t)X(0) = \int_{-\infty}^{+\infty} e^{iwt} d(E(w)X(0)) = e^{iHt}(X(0)) \tag{1}$$

here we have used the famous spectral Stone-theorem to re-write the associated time-translation unitary group in terms of the spectral process $dE(w)X(0)$, where H denotes the infinitesimal unitary group operator $U(t)$. We have supposed too that the σ-algebra generated by the $X(t)$-process is the whole measure space Ω, and $X(t)$ is a separable process.

Let us, thus, consider the following linear continuous functional on the Hilbert (complete) space $\{L^2(\Omega), d\mu(\lambda)\}$ - the space of the square integrable random variables on Ω

$$L(Y(\lambda)) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt E\{Y(\lambda)\overline{X(t,\lambda)}\}. \tag{2}$$

By a straightforward application of the R.A.G.E. theorem ([3]), namely:

$$L(Y(\lambda)) = \int_\Omega d\mu(\lambda)\, Y(\lambda) \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, e^{-iwt}\, \overline{dE(w)X_0(\lambda)} \right\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\{Y e^{-iHt}\overline{X}\} dt$$

$$\begin{aligned} &= \int_\Omega d\mu(\lambda) Y(\lambda) \overline{dE(0)X(0,\lambda)} \\ &= E\big\{Y(\lambda)\overline{P_{\mathrm{Ker}(H)}(X(0,\lambda))}\big\} \end{aligned} \tag{3}$$

Here $P_{\mathrm{Ker}(H)}$ is the (ortoghonal projection) on the kernel of the unitary-group infinitesimal generator H (see eq (1)).

By a straightforward application of the Riesz-representation theorem for linear functionals on Hilbert Spaces, one can see that $\overline{P_{\mathrm{Ker}(H)}(X(0))} d\mu(\lambda)$ is the searched time-independent ergodic-invariant measure associated to the ergodic theorem statement, i.e.: For any square integrable time independent random variable $Y(\lambda) \in L^2(\Omega, d\mu(\lambda))$, we have the ergodic result $(X(0,\lambda) = X(0)$.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, E\big\{X(t)\overline{Y}\big\} = E\big\{P_{\mathrm{Ker}(H)}(X(0))\overline{Y}\big\} \tag{4}$$

In general grounds, for any real bounded borelian function it is expected the result (not proved here)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, E\big\{f(X(t))\overline{Y}\big\} = E\big\{P_{\mathrm{Ker}(H)} f(X(0)) \cdot \overline{Y}\big\} \tag{5}$$

For the auto-correlation process function, we still have the result for the translated time ζ fixed (the lag time) as a direct consequence of eq(1) or the process' stationarity property

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\,E\big\{X(t)X(t+\zeta)\big\}=E\big\{X(0)X(\zeta)\big\} \tag{6}$$

It is important remark that we still have the probability average inside the ergodic time-averages eq(4)-eq(6). Let us call the reader attention that in order to have the usual Ergodic like theorem result - without the probability average E on the left-hand side of the formulae, we proceed [as it is usually done in probability text-books ([1])] by analyzing the probability convergence of the single sample stochastc-variables below [for instance]

$$\eta_T=\frac{1}{2T}\int_{-T}^{T}f(X(t))dt \tag{7}$$

$$R_T(\zeta)=\frac{1}{2T}\int_{-T}^{T}dt\,X(t)X(t+\zeta) \tag{8}$$

It is straightforward to show that if $E\big\{f(X(t))f(X(t+\zeta))\big\}$ is a bounded function of the time-lag, or, if the variance below written goes to zero at $T\to\infty$

$$\sigma_T^2=\lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{T}dt_1\int_{-T}^{T}dt_2\big[E\big\{X(t_1)X(t_1+\zeta)X(t_2)X(t_2+\zeta)\big)-$$
$$-E\big\{X(t_1)X(t_1+\zeta)\big\}E\big\{X(t_1)X(t_1+\zeta)\big)\big)\big]=0, \tag{9}$$

one has that the random variables as given by eq(7)-eq(8) converge at $T\to\infty$ to the left-hand side of eq(4)-eq(6) and producing thus an ergodic theorem on the equality of ensemble-probability average of the wide sense stationary process $\{X(t),-\infty<t<\infty\}$ and any of its single-sample $\{\overline{X}(t),-\infty<t<+\infty\}$ time average

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\,f(\overline{X}(t))=E\big\{P_{\mathrm{Ker}(H)}(f(X_0))\big\}=E\{f(X(t))\} \tag{10}$$

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\,\overline{X}(t)\overline{X}(t+\zeta)&=E\big\{X(0)X(\zeta)\big\}\\&=E\big\{X(t)X(t+\zeta)\big\}\\&=R_{XX}(\zeta)\end{aligned} \tag{11}$$

The above written formulae will be analyzed in next Electrical Engeneering oriented section.

3 A Sampling Theorem for Ergodic Process

Let us start this section by considering $F(w)$ as an entire complex-variable function such that

$$|F(w)| \leq c\, e^{A|w|}, \text{ for } w = x + iy \in \mathbb{C} = R + iR \tag{12}$$

and such its restriction to real domain $w = x + i0$ satisfies the square integrable condition

$$\int_{-\infty}^{\infty} |F(x)|^2 \, dx < \infty. \tag{13}$$

By the famous Wienner Theorem (ref. [4]) there exists a function $f(t) \in L^2(-A, A)$, vanishing outside the interval $[-A, A]$, such that

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} dt\, f(t) e^{itw} \tag{14}$$

The time-limited function $f(t)$ can be considered as a physically finite-time observed (time-series) of an ergodic process on the sample space $\Omega = L^2(R)$ as considered in section I (since $R_{XX}(0) < +\infty$). In a number of cases, one should use interpolation formulae in order to analyze several statistics aspects of such observed random process. One of the most useful result in this direction, however based in the very deep theorem of Carleson on the pontual convergence of Fourier series of a square integrable function ([1], [5]) is the well-known Shanon (in the frequency-domain) sampling theorem for time-limited signals, namely ([1], [6])

$$F(w) = \sum_{n=-\infty}^{+\infty} F\left(\frac{\pi n}{T}\right) \frac{\operatorname{sen}\left[T\big(w - \frac{\pi n}{T}\big)\right]}{T(\big(w - \frac{\pi n}{T}\big)} \tag{15}$$

Note in eq(15) however, the somewhat restrictive sampling condition hypothesis of the Nyquist interval condition $T \geq A$. Althought quite useful, there are situations where its direct use may be cumbersome due to the Nyquist condition on the signal sampling besides the explicitly necessity of infinite time observations $\{\frac{\pi n}{T}\}_{n \in \mathbb{Z}}$.

At this point, let us propose the following more invariant sampling - interpolation result of ours. Let $f(t)$ be an observed time-finite Random ergodic signal (sample) as mathematically considered in eq(12)-eq(14). It is well-known that we have the famous uniform convergent

Fourier Expansion for $t = A \operatorname{sen}\theta$, with $-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}$ for the Fourier Kernel in terms of Bessel functions

$$e^{+iwt} = e^{+iw(A \operatorname{sen}\theta)} = \sum_{n=-\infty}^{+\infty} J_n(wA)e^{in\theta} \tag{16}$$

As much as in the usual proof of the Shanon results eq(15), we introduce eq(16) into eq(14) and by using the Lebesgue convergence theorem, since $f(t) \in L^1(-A, A)$ either, we get the somewhat (canonical) interpolating formula without any Nyquist - like restrictive sampling frequency condition on the periodogran $F(w)$

$$F(w) = \sum_{n=-\infty}^{+\infty} AJ_n(wA) \left\{ \int_{-\pi/2}^{\pi/2} d\theta\, e^{in\theta}\, f(A \operatorname{sen}\theta) \cos\theta \right\} \tag{17}$$

It is worth call attention that the sampling coefficients as given by eq(17) are exactly the values of the Fourier transform of the signal $g(\theta) = f(A \operatorname{sen}\theta)\cos\theta$ for $-\frac{\pi}{2} < \theta < \frac{\theta}{2}$, i.e.:

$$\begin{aligned} g_n &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\theta\, e^{in\theta}\, f(A \operatorname{sen}\theta)\cos\theta \qquad (18) \\ &= \frac{1}{2\pi} \int_{-A}^{A} dt\, e^{\text{in arc sin}(\frac{t}{A})}\, f(t) \simeq \sum_{i=1}^{N} \left(\frac{1}{2\pi} \exp(\text{in arc}\operatorname{sen}(\frac{t_i}{A})) \cdot f(t_i) \right) \end{aligned}$$

which may be easily and straightforwardly evaluated by FFT algorithms, from arbitrary - finite on their number - chosen sampling values $\{f(A \operatorname{sen}\theta_i)\}_{i=1,\ldots,N}$ with $\theta_i = arsen\left(\frac{t_i}{A}\right)$ [here t_i are the time observation process signal]. Monte-Carlo integration techniques ([7]) can be used on approximated evaluations of eq.(18) too. Note that $f(w)$ is completely determined by its samples eq(18) taken at arbitrary times $t_i \in [-A, A]$.

In the general case of a quadratic mean continuous wide-sense stationary process $\{X_t, 0 < t < \infty\}$ ([1]), we still have the quadratic mean result analogous to the above exposed result for those process with time-finite sampling ([1])

$$X_t = \int_{-\infty}^{+\infty} e^{iwt}\, d\widehat{X}_w \tag{19}$$

with the spectral process possesing the canonical form (in the quadratic mean sense)

$$\widehat{X}(w) = \sum_{n=-\infty}^{+\infty} J_n(wA) \left\{ \int_{-\pi/2}^{\pi/2} e^{in\theta}\, d(X_{t=A \operatorname{sen}\theta}) \right\} \tag{20}$$

Let us expose the usefulness of the sampling result eqs(17)-(18) for the very important practical engineering problem of estimate the self-correlation function of a time-finite observed sampling of a Ergodic process already taking into consideration the time-limitation of the sampling observation in the estimate formulae. We have, thus, to evaluate the (formal) time-average with time-lag $\zeta > 0$

$$R_{ff}(\zeta) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, f(t) f(t+\zeta). \tag{21}$$

Since we have observed the sampling-continuous function $\{f(t)\}$ only for the time-limited observation interval ($[-A, A]$), it apears quite convenient at this point to re-write eq(21) in the frequency domain (as a somewhat generalized process) (see [1]) where the $T \to \infty$ limit is already evaluated

$$\begin{aligned} R_{ff}(\zeta) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dw_1 dw_2\, e^{iw_1\zeta} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, e^{it(w_1+w_2)} F(w_1) F(w_2) \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dw |F(w)|^2\, e^{iw\zeta} \end{aligned} \tag{22}$$

It is worth point out that eq(22) has already "built" in its structure, the infinite time-ergodic evalution $\left\{ \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (...) \right\}$, regardless the time-limited finite duration nature of the observed sample $f(t)[\theta(t+A) - \theta(t-A)]$.

Now an expression for eq(22), after substituting eq(17) on eq(22) [without the use of somewhat artificial aliased A-periodic extensions of the observed sampling $f(t)$, as it is commonly used in the literature ([6])], can straightforwardly be suggested possesing the important property of already taking into account (in an explicitly way) the presence of the observation time A in the result for the self-correlation function in the frequency domain

$$S_{ff}(w) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left\{ \left[A^2 (g_n \bar{g}_m) \right] \times J_n(wA) J_m(wA) \right\} \tag{23}$$

with

$$R_{ff}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dw\, e^{iw\zeta}\, S_{ff}(w) \tag{24}$$

The above written equations are the main results of this Section 3.

It is worth call the reader attention that after passing the random signal $f(t)$ by a causal linear system with transference function $H_{ff}(w)$ ([6]), one obtain the output self-correlation function statistics in the standard formulae

$$S_{yy}(w) = |H_{yf}(w)|^2 \times S_{ff} \tag{24}$$

Finally, let us comment on the evaluation of eq(23) on the time-domain. This step can be implemented throught the use of the well-known formula for Bessel functions (see [8] - eq 6.626), analytically continued in the relevant parameters formulae

$$\begin{aligned}
&\int_0^\infty dx\, e^{-ax}\, J_\mu(bx) J_\nu(cx) dx = I_{\mu,\nu}(a) \\
&= \frac{b^\mu c^\nu}{\Gamma(\nu+1)}\, 2^{-\mu-\nu}\, (a)^{-1-\mu-\nu} \Bigg\{ \sum_{\ell=0}^\infty \frac{\Gamma(1+\mu+\nu+2\ell)}{\ell!\Gamma(\mu+\ell+1)} \\
&\qquad \times F\left(-\ell, -\mu-\ell, \nu+1, \frac{c}{b^2}\right) \left(-\frac{b^2}{4a^2}\right)^\ell \Bigg\}
\end{aligned} \tag{25}$$

We have, thus, the result:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dw\, e^{iwt}\, J_n(Aw) \cdot J_m(wA) = \left[(-1)^{n+m}\, I_{n,m}(-\frac{t}{A}) + I_{n,m}(\frac{t}{A})\right] \tag{26}$$

where

$$\begin{aligned}
I_{n,m}(t) &= \frac{1}{\Gamma(m+1)}\, 2^{-m-n} (-it)^{-1-m-n} \\
&\times \left\{ \sum_{\ell=0}^\infty \frac{\Gamma(1+n+m+\ell)}{\ell!\Gamma(n+\ell+1)}\, F\left(-\ell, -n-\ell, m+1, 1\right) \left(-\frac{1}{4t^2}\right)^\ell \right\}
\end{aligned} \tag{27}$$

Another point to be called the attention and related to the integral evaluations of Fourier Transformed of Bessel Functions are the recurrence set of Fourier Integrals of Bessel functions below

$$\begin{aligned}
\int_{-\infty}^{+\infty} dt\, e^{iwt}\, J_n(w) &= (1+(-1)^n) \left\{ \int_0^\infty dt \cos(wt) J_n(w) \right\} \\
&\quad + i(1+(-1)^{n+1}) \left\{ \int_0^\infty dt\ \mathrm{sen}(wt) J_n(w) \right\}
\end{aligned} \tag{28}$$

with here, the explicitly expressions $T_n(t) = 2^{-n}[(t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n]$.

$$\int_0^\infty \cos(wt)J_n(w)dw = \begin{cases} (-1)^k \frac{1}{\sqrt{1-t^2}}T_n(t), \text{ for } 0<t<1 \text{ and } n=2k \\ 0, \text{for } 0<1<t \\ \int_0^\infty \cos(wt)\frac{4k}{w}J_{n-1}(w)dw - \int_0^\infty \cos(wt)J_{n-2}(w)dw \text{ for } n=2k+1 \end{cases} \tag{29}$$

$$\int_0^\infty \text{sen}(wt)J_n(w)dw = \begin{cases} (-1)^k \frac{1}{\sqrt{1-t^2}}T_n(t), \text{ for } 0<t<1 \text{ and } n=2k+1 \\ 0, \text{for } 0<1<t \\ 2(2k+1)\int_0^\infty \frac{\text{sen}(wt)}{w}J_{n-1}(w) - \int_0^\infty \text{sen}(wt)J_{n-2}(w)dw \text{ for } n=2k+2 \end{cases} \tag{30}$$

4 A Model for the Turbulent Pressure Fluctuations (Random Vibrations Transmission)

One of the most important studies of pressure turbulent fluctuations (random vibrations) is to estimate the turbulent pressure component transmission inside fluids ([9], [10]). In this section we intend to propose a simple analysis of a linear model of such random pressure vibrations.

Let us consider a infinite beam backed on the lower side by a space of depth d which is filled with a fluid of density ρ_2 and sound speed v_2. On the upper side of the beam there is a supersonic boundary-layer turbulent pressure $P(x,t)$. The fluid on the upper side of the beam which is on the turbulence steadily regime is supposed to have a free stream velocity U_∞, density ρ_1 and sound speed v_1.

The effective equation governing the "outside" pressure in our model is thus given by ($d \le z < \infty$)

$$\left(\frac{\partial}{dt} + U_\infty \frac{\partial}{\partial x}\right)^2 p_1(x,z,t) - v_1^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 z}\right)p_1(x,z,t) = 0 \tag{31}$$

with the boundary condition (2^{nd} Newton's law)

$$-\rho_1\left(\frac{\partial}{\partial t} + U_\infty\frac{\partial}{\partial x}\right)^2 W(x,t) = \frac{\partial p_1(x,z,t)}{\partial z}\Big|_{z=d}. \tag{32}$$

Here the beam's deflection $W(x,t)$ is assumed to be given by the beam small linear deflection equation

$$B\frac{\partial^4 W(x,t)}{\partial^4 x} + m\frac{\partial^2 W(x,t)}{\partial^2 t} = P(x,t) + (p_1 - p_2)(x,d,t) \tag{33}$$

with B denoting the bending rigidity, m the mass per unit lenght of the beam and $p(x,z,t)$ the (supersonic) boundary-layer pressure fluctuation of the exterior medium.

The searched induced pressure $p_2(x,z,t)$ in the fluid interior medium $(0 \le z \le d)$ is governed by

$$\frac{\partial^2 p_2}{\partial^2 t}(x,z,t) - (v_2)^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 z}\right)p_2(x,z,t) = 0 \tag{34}$$

$$\frac{\partial p_2(x,d,t)}{\partial z} = -\rho_2\frac{\partial^2}{\partial^2 t}W(x,t) \tag{35}$$

The solution of eq.(31) and eq.(32) is straightforward obtained in the Fourier domain

$$\begin{aligned}\tilde{p}_1(k,z,w) = -\rho_1&\left\{\frac{v_1(w+U_\infty k)^2}{\sqrt{v_1^2k^2-(w+U_\infty k)^2}}\right\}\tilde{w}(k,w)\\ \times \exp&\left(-\frac{1}{v_1}\times\sqrt{v_1^2k^2-(w+U_\infty k)^2}\right)(z-d)\end{aligned} \tag{36}$$

where the deflection beam $\tilde{W}(k,w)$ is explicitly given by

$$\begin{aligned}\tilde{W}(k,w) &= (\tilde{P}(k,w) - \tilde{p}_2(k,w,d)\\ \times&\left\{Bk^4 - mw^2 + \frac{\rho_1(w+U_\infty k)^2 v_1}{\sqrt{v_1^2k^2-(w+U_\infty\cdot k)^2}}\right\}^{-1}.\end{aligned} \tag{37}$$

At this point, we solve our problem of determining the pressure $\tilde{p}_2(k,w,z)$ in the interior domain eq.(34)–eq.(35) if one knows the pressure $\tilde{p}_2(k,w,d)$ on the boundary $z = d$. Let us, thus, consider the Taylor's serie in the z-variable $(0 \le z \le d)$ around $z = d$, namely

$$\tilde{p}_2(k,w,z) = \tilde{p}_2(k,w,d) + \frac{\partial\tilde{p}_2(k,w,z)}{\partial z}\Big|_{z=d}(z-d) + \cdots + \frac{1}{k!}\frac{\partial^k\tilde{p}_2(k,w,z)}{\partial^k z}(z-d)^k + \cdots \tag{38}$$

From the boundary condition eq.(35), we have the explicitly expression for the second-derivative on the depth z

$$\begin{aligned}\frac{\partial\tilde{p}_2(k,w,z)}{\partial z}\Big|_{z=d} &= +\rho_2 w^2\tilde{W}(k,w) =\\ +\rho_2 w^2[\tilde{P}(k,w) - \tilde{p}_2(k,w,d)]&\times\left\{Bk^4 - mw^2 + \frac{\rho_1(w+U_\infty k)^2 v_1}{\sqrt{v_1k^2-(w+U_\infty k)^2}}\right\}^{-1}\end{aligned} \tag{39}$$

The second z-derivative of the interior pressure (and the higher ones!) are easily obtained recursively from the wave equation (34) ($k \geq 0,\ k \in \mathbb{Z}^{+}$) and eq(39)

$$\frac{\partial^{2+k}}{\partial^{2+k} z}\tilde{p}_2(k,z,w)\Big|_{z=d} = \left(\frac{1}{(v_2)^2} \times (-w^2 + (v_2)^2 k^2)\right)\left(\frac{\partial^k}{\partial^k z}\tilde{p}_2(k,z,w)\right)\Big|_{z=d}. \qquad (40)$$

Let us finally make the connection of this random transmission vibration model with the section 3 by calling the reader attention that the general turbulent pressure is always assumed to be expressible in a integral form

$$P(x,t) = \int_{-\infty}^{+\infty} dw \int_{-\infty}^{+\infty} dk\ e^{ikx} e^{i(w-ku)t} G(k) \cdot F(w) \qquad (41)$$

where $F(w)$ is the Fourier Transform of a time-limited finite duration sample function of an Ergodic Process ([1]) simulating the stochastic-turbulent nature of the pressure field acting on the fluid with the exactly interpolating formulae eq.(17)-eq.(18). Numerical studies of this random vibration transmission will be presented elsewhere.

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